

Surfaces in three-dimensional Lie groups in terms of spinors

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Recently surfaces in three-dimensional homogeneous spaces which differ from the space forms attract a lot of attention. Mainly for ambient spaces there are taken three-dimensional spaces with the Thurston geometries ¹ or simply-connected spaces with a four-dimensional isometry group. ²

We consider the case when the ambient space is a Lie group because it is straightforward (see [4]) to generalize the Weierstrass representation of surfaces in \mathbb{R}^3 to this case. This representation involves the Dirac operator which plays an important role in many integrable soliton equations and has a rich and far-developed spectral theory [21, 22].

In particular, we have been interested from the beginning in the following questions:

1) it is known that certain classes of surfaces in the space forms are described by some integrable systems (for instance, constant mean curvature tori).

How such surfaces are described in new geometries?

If these surfaces are described by some integrable systems how these systems obtained from the old ones and how the curvature of the ambient space contribute to the deformation of an integrable system?

2) it is known that some spectral data of the Dirac operator coming in the Weierstrass representation of a surface in \mathbb{R}^3 have geometrical meanings and, in particular, the Willmore functional serves as an example [21, 22].

What mean these spectral data for surfaces in other ambient spaces (Lie groups)?

We discuss some partial answers to these questions in §§4 and 5.

We also would like to mention that the choice of Lie groups as the ambient spaces is not very restrictive since it covers all spaces $E(\kappa, \tau)$ but $S^2 \times \mathbb{R}$ and

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¹These are the space forms \mathbb{R}^3 , S^3 , and \mathcal{H}^3 ; the product geometries $S^2 \times \mathbb{R}$ and $\mathcal{H}^2 \times \mathbb{R}$; and three geometries modeled on the Lie groups Nil, Sol, and $SL(2, \mathbb{R})$ with certain left-invariant metrics.

² All such spaces are locally isometric to line bundles (with the bundle curvature τ) over space forms (with the curvature κ): for $\kappa \neq 4\tau^2$ we have the spaces $E(\kappa, \tau)$ from the table below

	$\kappa < 0$	$\kappa = 0$	$\kappa > 0$
$\tau = 0$	$\mathcal{H}^2 \times \mathbb{R}$	\mathbb{R}^3	$S^2 \times \mathbb{R}$
$\tau \neq 0$	$\widetilde{SL(2, \mathbb{R})}$	Nil	Berger spheres

and for $\kappa = 4\tau^2$ we have spaces of constant curvature.

the Thurston geometries again except $S^2 \times \mathbb{R}$ (see remarks on page 9).

1. The Weierstrass representation of surfaces in \mathbb{R}^3 and the Willmore functional

The original Weierstrass representation of minimal surfaces in \mathbb{R}^3 may be considered as an integrable system in geometry because it gives an explicit formula for a general solution to the minimal surface equation in \mathbb{R}^3 in terms of a pair of arbitrary holomorphic functions. It is as follows. Let $z \in D \subset \mathbb{C}$ and, for simplicity, assume that a domain D is simply connected. Let f and g be holomorphic functions on D . Then the Weierstrass (–Enneper) formulas

$$\begin{aligned} x^1(z, \bar{z}) &= x_0^1 + \frac{i}{2} \int [(f^2 + g^2)dz - (\bar{f}^2 + \bar{g}^2)d\bar{z}], \\ x^2 &= x_0^2 + \frac{1}{2} \int [(g^2 - f^2)dz + (\bar{g}^2 - \bar{f}^2)d\bar{z}], \\ x^3 &= x_0^3 + \int (fgdz + \bar{f}\bar{g}d\bar{z}) \end{aligned} \quad (1)$$

define a minimal surface in \mathbb{R}^3 . Here the integrals defining $x(P)$, the image of $P \in D$, are taken along a path $\gamma \subset D$ from the point P_0 such that $x(P_0) = x_0$ to P . Since the integrands are closed forms this is independent on the choice of γ . The induced metric takes the form $(|f|^2 + |g|^2)^2 dz d\bar{z}$ and therefore z is a conformal parameter on the surface

In fact, the condition that z is a conformal parameter is written as

$$\left(\frac{\partial x^1}{\partial z}\right)^2 + \left(\frac{\partial x^2}{\partial z}\right)^2 + \left(\frac{\partial x^3}{\partial z}\right)^2 = 0,$$

i.e., $(r_u, r_u) = (r_v, r_v)$, $(r_u, r_v) = 0$ where u and v are the isothermic coordinates such that $z = u + iv$, and $r_u = 2\text{Re}\frac{\partial x}{\partial z}$ and $r_v = -2\text{Im}\frac{\partial x}{\partial z}$ are the corresponding tangent vectors to the surface. The quadric

$$Q = \{y_1^2 + y_2^2 + y_3^2 = 0\} \subset \mathbb{C}P^2$$

gives a one-to-one parametrization of oriented two-planes in \mathbb{R}^3 by corresponding to every plane its homogeneous coordinates $((\xi^1 - i\eta^1) : (\xi^2 - i\eta^2) : (\xi^3 - i\eta^3))$ where (ξ, η) is a positively oriented basis for the plane such that $|\xi| = |\eta|$ and ξ is orthogonal to η . Due to the homogeneity of coordinates in $\mathbb{C}P^2$ this mapping is correctly defined, i.e., is independent on the choice of a basis (ξ, η) . Hence the mapping

$$P \rightarrow \left(\frac{\partial x^1(P)}{\partial z} : \frac{\partial x^2(P)}{\partial z} : \frac{\partial x^3(P)}{\partial z}\right) \in Q$$

is the Gauss map of the surface. The quadric Q , the Grassmannian of oriented two-planes in \mathbb{R}^3 , admits a natural rational parametrization:

$$(f : g) \rightarrow \left(\frac{i}{2}(f^2 + g^2) : \frac{1}{2}(g^2 - f^2) : fg\right). \quad (2)$$

From this interpretation of the Gauss map it is clear that

- any surface, not only minimal, is defined by the Weierstrass formulas for the factorization (f, g) of the Gauss map.

The Gauss–Codazzi equations written in terms of (f, g) distinguish mappings $D \xrightarrow{(f,g)} Q$ which are the Gauss maps of surfaces. It is straightforward to compute that these equations take the form

$$\mathcal{D}\psi = 0$$

where \mathcal{D} is the Dirac operator

$$\mathcal{D} = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \quad (3)$$

and

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \begin{pmatrix} f \\ \bar{g} \end{pmatrix}.$$

For surfaces in \mathbb{R}^3 the potentials U and V and the induced metric are

$$U = V = \frac{He^\alpha}{2}, \quad e^{2\alpha}dzd\bar{z} = (|\psi_1|^2 + |\psi_2|^2)^2dzd\bar{z}. \quad (4)$$

We conclude that

- a general surface in \mathbb{R}^3 is represented by the Weierstrass formulas (1) for some solution to the Dirac equation with the potentials (4) and the inverse is also true: any solution to the Dirac equation with real-valued potentials $U = V$ defines via (1) a surface in \mathbb{R}^3 with the mean curvature and the induced metric given by (4).

This representation has some prehistory for which we refer to [22] however for $U \neq 0$ the formulas in terms of the Dirac operator first appeared in [15] where they were introduced for inducing surfaces admitting certain soliton deformations. This operator has a rich spectral theory and, in particular, we started in [21] to study possible relations between the spectral properties of \mathcal{D} and the geometry of the corresponding surfaces. In particular, it appears that for a closed oriented surface $M \subset \mathbb{R}^3$ the integral

$$E(M) = \int_M UV dx dy \quad (5)$$

is one-fourth of the Willmore functional

$$\mathcal{W}(M) = \int_M H^2 d\mu \quad (6)$$

where $d\mu$ is the induced measure on M . The Willmore functional is the basic functional in the conformal surface geometry, and the integral (5) is an important spectral quantity of the Dirac operator \mathcal{D} .

The Willmore conjecture states that \mathcal{W} attains its minima for tori which is equal to $2\pi^2$ on the Clifford torus and its images under conformal transformations of \mathbb{R}^3 . The existence of the lower bounds for \mathcal{W} on closed surfaces is explained by the Weierstrass representation as follows:

- there are no compact minimal surfaces without boundary in \mathbb{R}^3 . We have to perturb the potential U from the zero level to achieve compact surfaces and the threshold for the L_2 -norm of U at which compact surfaces appear gives this minimum level. For surfaces in \mathbb{R}^3 we have $U = \bar{U} = V$, the energy (5) is the squared L_2 -norm of U and it is also one-fourth of \mathcal{W} .

We propose an approach to the Willmore conjecture based on the spectral properties of the corresponding double-periodic (for tori) Dirac operator. Several attempts to realize this approach led to interesting results however the conjecture stays open until recently. We refer for the survey of the Willmore conjecture and the spectral approach to its study to [22].

The classical Weierstrass representation for minimal surfaces corresponds to the case $U = 0$ and it enables us to consider the minimal surface equation in \mathbb{R}^3 as an integrable system. The integrability property resolves the local theory and does not help straightforwardly in answering questions on the global behavior of surfaces. The global theory needs an additional technique concerning global solutions to the integrable system (in the case of minimal surfaces, holomorphic functions).

2. The Weierstrass representation of surfaces in three-dimensional Lie groups [4]

To generalize the Weierstrass representation for the case when the ambient space is a three-dimensional Lie group G with a left-invariant metric [4] we have to replace $\frac{\partial x}{\partial z} \in \mathbb{C}^3$ by the element of the complexified Lie algebra:

$$\frac{\partial}{\partial z} \in \mathbb{C}^3 \longrightarrow \Psi = f^{-1} \frac{\partial f}{\partial z} \in \mathfrak{g} \otimes \mathbb{C}$$

where

$$f : M \rightarrow G$$

is an immersion of a surface and z is a conformal parameter on M . In terms of Ψ and $\Psi^* = f^{-1} f_{\bar{z}} \bar{\Psi}$ the derivational equations take the form

$$\partial \Psi^* - \bar{\partial} \Psi + \nabla_{\Psi} \Psi^* - \nabla_{\Psi^*} \Psi = 0,$$

$$\partial \Psi^* + \bar{\partial} \Psi + \nabla_{\Psi} \Psi^* + \nabla_{\Psi^*} \Psi = e^{2\alpha} H f^{-1}(N)$$

where the Levi-Civita connection on G is linearly expanded onto complex-valued vectors Ψ and Ψ^* , N is the unit normal vector field to M and $e^{2\alpha} dz d\bar{z}$ is the induced metric. Originally these equations were first derived for minimal surfaces in [11].

Given an orthonormal basis e_1, e_2, e_3 for \mathfrak{g} , we expand Ψ in this basis

$$\Psi = Z_1 e_1 + Z_2 e_2 + Z_3 e_3.$$

The conformality condition again takes the form

$$Z_1^2 + Z_2^2 + Z_3^2 = 0.$$

Let us use the same factorization of $Z : M \rightarrow Q$ as in the Euclidean case:

$$Z_1 = \frac{i}{2}(\bar{\psi}_2^2 + \psi_1^2), \quad Z_2 = \frac{1}{2}(\bar{\psi}_2^2 - \psi_1^2), \quad Z_3 = \psi_1 \bar{\psi}_2.$$

The derivational equations take the form of the Dirac equation

$$\mathcal{D}\psi = 0$$

and the induced metric is again equal to

$$e^{2\alpha} dz d\bar{z} = (|\psi_1|^2 + |\psi_2|^2)^2 dz d\bar{z}.$$

Therewith we call ψ a generating spinor of a surface.

In difference with the Euclidean case, the potentials U and V are not always real-valued and do not always coincide.

Theorem 1 ([4]) *The potentials of the Weierstrass representation of surfaces in the Lie groups $SU(2)$, Nil , $\widetilde{SL(2, \mathbb{R})}$, and Sol , endowed with the Thurston geometries, are as follows:*

1. $G = SU(2)$:

$$U = \bar{V} = \frac{1}{2}(H - i)e^\alpha;$$

2. $G = \text{Nil}$:

$$U = V = \frac{He^\alpha}{2} + \frac{i}{4}(|\psi_2|^2 - |\psi_1|^2);$$

3. $G = \widetilde{SL(2, \mathbb{R})}$:

$$U = \frac{He^\alpha}{2} + i \left(\frac{1}{2}|\psi_1|^2 - \frac{3}{4}|\psi_2|^2 \right), \quad V = \frac{He^\alpha}{2} + i \left(\frac{3}{4}|\psi_1|^2 - \frac{1}{2}|\psi_2|^2 \right);$$

4. $G = \text{Sol}$:³

$$U = \frac{He^\alpha}{2} - \frac{1}{2}\bar{\psi}_2^2 \frac{\bar{\psi}_1}{\psi_1}, \quad V = \frac{1}{2}He^\alpha + \frac{1}{2}\bar{\psi}_1^2 \frac{\bar{\psi}_2}{\psi_2}.$$

These potentials are written with respect to certain choices of orthogonal bases for \mathfrak{g} which are as follows:

- a) Sol admits a natural splitting

$$1 \rightarrow \mathbb{R}^2 \rightarrow \text{Sol} \rightarrow \mathbb{R}$$

which induces the submersion $\text{Sol} \rightarrow \mathbb{R} = \text{Sol}/\mathbb{R}^2$ whose leaves are minimal surfaces. We put e_3 to be the pullback of the unit vector on \mathbb{R} . Hence, $Z_3 = \psi_1 \bar{\psi}_2 = 0$ if the tangent plane to a surface is tangent to a minimal leave. For

³Here we correct the sign of the second term in the expression for U miscalculated in [4].

a surface in Sol the Dirac equation is correctly defined only in domain $D = \{Z_3 \neq 0\}$. It is natural to assume that $U = V = 0$ outside D . Then the Dirac equations hold everywhere outside ∂D , the boundary of D , at which $\frac{\bar{\psi}_1}{\psi_1}$ and $\frac{\bar{\psi}_2}{\psi_2}$ may have indeterminacies;

b) for Nil and $\widetilde{SL(2, \mathbb{R})}$ we assume that e_3 is directed along the axis of isometry rotation. Both these groups admit four-dimensional isometry groups and such an axis is uniquely defined everywhere.

These Dirac equations differ from their Euclidean analog in several aspects:

a) there are constraints which relate solutions ψ corresponding to surfaces with potentials. In the Euclidean case any solution corresponds to a surface. This demonstrates the absence of dilations in these Lie groups;

b) the reconstruction of the surface $f : M \rightarrow G$ from ψ needs solving the linear equation

$$f_z = f\Psi.$$

In the Euclidean case a solution to this equation is given by (1);

c) solutions to these Dirac equation does not admit the quaternion symmetry, i.e., if $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ satisfies $\mathcal{D}\psi = 0$ then in general $\psi^* = \begin{pmatrix} -\bar{\psi}_2 \\ \bar{\psi}_1 \end{pmatrix}$ does not meet this equation. This hinders to use the Dirac equation for interpreting surfaces as holomorphic sections of certain line bundles and applying some ideas of algebraic geometry as it is done for surfaces in \mathbb{R}^3 and \mathbb{R}^4 in [8].

Corollary 1⁴ *The generating spinors of minimal surfaces in the Lie groups Nil, $\widetilde{SL(2, \mathbb{R})}$, and Sol are given by the following equations:*

1. $G = \text{Nil}$:

$$\bar{\partial}\psi_1 = \frac{i}{4}(|\psi_2|^2 - |\psi_1|^2)\psi_1, \quad \partial\psi_2 = -\frac{i}{4}(|\psi_2|^2 - |\psi_1|^2)\psi_2;$$

2. $G = \widetilde{SL(2, \mathbb{R})}$:

$$\bar{\partial}\psi_1 i \left(\frac{3}{4}|\psi_1|^2 - \frac{1}{2}|\psi_2|^2 \right) \psi_2, \quad \partial\psi_2 - i \left(\frac{1}{2}|\psi_1|^2 - \frac{3}{4}|\psi_2|^2 \right) \psi_1;$$

3. $G = \text{Sol}$:

$$\bar{\partial}\psi_1 = \frac{1}{2}\bar{\psi}_1^2\bar{\psi}_2, \quad \partial\psi_2 = \frac{1}{2}\bar{\psi}_1\bar{\psi}_2^2.$$

In other terms the Weierstrass type representations for minimal surfaces in Nil and Sol were derived in [12, 13].

We remark that Friedrich showed that the ψ -spinor for surfaces in \mathbb{R}^3 may be interpreted as the restriction of the parallel spinor field in \mathbb{R}^3 onto the surface [10]. Later a similar description of such representations for surfaces in S^3 and \mathcal{H}^3 was derived in [16] and very recently the same was done for surfaces in the

⁴We skip here the well-studied case of minimal surfaces in the unit three-sphere $SU(2)$.

spaces with a four-dimensional isometry group [20] (this paper uses description of immersions in other terms obtained in [6]). In the first case the parallel spinor field is replaced by real and imaginary Killing fields and in the second case it is replaced by certain generalized Killing spinor fields.

3. Surfaces in general Lie groups and families of Lie groups.

The Weierstrass representation method admits us to write such representations straightforwardly for a general Lie group and even to consider surfaces in families of Lie groups. We demonstrate that for a certain family which includes some well-known spaces.

Let us remind THE BIANCI CLASSIFICATION of real three-dimensional Lie algebras.

For such an algebra \mathfrak{g} there is a basis e_1, e_2, e_3 such that the commutation relations takes the form

$$[e_1, e_2] = ae_2 + b^{(3)}e_3, \quad [e_1, e_3] = ae_3 - b^{(2)}e_2, \quad [e_2, e_3] = b^{(1)}e_1$$

with $ab^{(1)} = 0$, hence the Lie algebra is included in the following table

Type	a	$b^{(1)}$	$b^{(2)}$	$b^{(3)}$	Type	a	$b^{(1)}$	$b^{(2)}$	$b^{(3)}$
I	0	0	0	0	VI_0	0	1	-1	0
II	0	1	0	0	$\text{VI}_a, 0 < a < \infty, a \neq 1$	a	0	1	-1
III	1	0	1	-1	VII_0	0	1	1	0
IV	1	0	0	1	$\text{VII}_a, a > 0$	a	0	1	1
V	1	0	0	0	VIII	0	1	1	-1
					IX	0	1	1	1

and algebras corresponding to different entries of this table are pairwise non-isomorphic.

The simply-connected Lie groups with Lie algebras of types I–VII have the form

$$1 \rightarrow \mathbb{R}^2 = H \rightarrow G \rightarrow G/H = \mathbb{R} \rightarrow 1 \quad (7)$$

and such an extension is uniquely defined by the action

$$\text{Ad}_z X = zXz^{-1} = e^{Az}X, \quad z \in G/H, \quad X = \begin{pmatrix} x \\ y \end{pmatrix} \in H, \quad A \in \mathfrak{gl}(2, \mathbb{R}).$$

In terms of Lie algebras we have

$$\text{ad}_\eta \xi = [\eta, \xi] = A\xi$$

where η and the Lie algebra \mathfrak{h} of H span \mathfrak{g} and $\xi \in \mathfrak{h}$. The matrices A and λBAB^{-1} , $\lambda = \text{const} \neq 0$, define isomorphic extensions.

We have

I: $G = \mathbb{R}^3$, $A = 0$.

II: $G = \text{Nil}$, the nilpotent group, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

III: $G = \mathbb{R} \times A(1)$, where $A(1)$ is the group of all affine transformations of \mathbb{R}^1 ; $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

IV: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

V: $G = A(2)$, the group formed by three-dimensional affine transformations of the form

$$\begin{pmatrix} e^t \cdot I_2 & s \\ 0 & 1 \end{pmatrix}, \quad t \in \mathbb{R}, s \in \mathbb{R}^2; \quad (8)$$

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, I_2 is the unit (2×2) -matrix.

VI₀: $G = \text{Sol}$, the solvable group; $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

VI _{$a, a \neq 0$} : $A = \begin{pmatrix} a & -1 \\ -1 & a \end{pmatrix}$, the eigenvalues $\lambda_{1,2}$ of A are $\lambda_{1,2} = a \pm 1$.

VII₀: $G = E(2)$, the group of all isometries of \mathbb{R}^2 ; $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

VII _{$a, a \neq 0$} : $A = \begin{pmatrix} a & 1 \\ -1 & a \end{pmatrix}$, the eigenvalues of A are $\lambda_{1,2} = a \pm i$.

The algebras of types VIII and IX do not contain two-dimensional commutative subalgebras and hence does not admit the representation (7). We have

VIII: $G = \widetilde{SL(2, \mathbb{R})}$, the universal cover of $SL(2, \mathbb{R})$, which is also locally isomorphic to $SO(1, 2)$ and $SU(1, 1)$.

IX: $G = SU(2) = \widetilde{SO(3)}$.

A left-invariant metric on a Lie group G is uniquely defined by its value at the unit of G , i.e. by an inner product on the Lie algebra \mathfrak{g} . Given an orthonormal basis e_1, \dots, e_n for \mathfrak{g} : $\langle e_i, e_j \rangle = \delta_{ij}$, $i, j = 1, \dots, n$, we denote by the same symbols the corresponding left-invariant vector fields. The Levi-Civita connection is given by the following formulas:

$$\nabla_{e_k} e_j = \Gamma_{jk}^i e_i, \quad \Gamma_{jk}^i = \frac{1}{2} (c_{kj}^i + c_{ik}^j + c_{ij}^k), \quad [e_i, e_j] = c_{ij}^k e_k.$$

Let us denote by H_n the group of all n -dimensional affine transformations of the form (8) with $s \in \mathbb{R}^{n-1}$. By simple computations we obtain

Proposition 1⁵ *Let us endow the group H_n by the left-invariant metric for which $e_1 = \frac{\partial}{\partial t}$, $e_2 = \frac{\partial}{\partial s^1}$, \dots , $e_n = \frac{\partial}{\partial s^{n-1}}$ for the orthonormal basis in \mathfrak{g} . Then H_n is isometric to the n -dimensional hyperbolic space \mathcal{H}^n .*

Corollary 2 *The group of type III with a certain left-invariant metric is isometric to $\mathcal{H}^2 \times \mathbb{R}$.*

⁵Recently we have known that such a representation of the hyperbolic three-space was used by Kokubu for deriving the Weierstrass representation of minimal surfaces in \mathcal{H}^3 [14].

Corollary 3 *There is a left-invariant metric on the group of type V such that such a Riemannian manifold is isometric to \mathcal{H}^n .*

H_n acts isometrically by left translations on $\mathcal{H}^n = \{(x, y), x \in \mathbb{R}^{n-1}, y \in \mathbb{R}, y > 0\}$ with the metric $\frac{dx^2 + dy^2}{y^2}$ as follows: $(x, y) \rightarrow (e^t x + s, e^t y)$.

We see that

- all simply-connected homogeneous three-spaces with a four-dimensional isometry group except $S^2 \times \mathbb{R}$ are isometric to Lie groups with left-invariant metrics
- all Thurston geometries except $S^2 \times \mathbb{R}$ are modeled by Lie groups with left-invariant metrics.

Let us consider the μ -parameter family G_μ of Lie groups of type (7) for which

$$A_\mu = \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix}.$$

For $-1 \leq \mu \leq 1$ these groups are pairwise nonisomorphic and as follows:

- $\mu = -1$: Sol, i.e. of the type VI₀;
- $-1 < \mu < 0$: VI_a, $0 < a < 1$, $\mu = \frac{a-1}{a+1}$;
- $\mu = 0$: III;
- $0 < \mu < 1$: VI_a, $1 < a < \infty$, $\mu = \frac{a-1}{a+1}$;
- $\mu = 1$: V.

Let us take the orthonormal basis e_1, e_2, e_3 such that

$$[e_1, e_2] = 0, \quad [e_3, e_1] = \mu e_1, \quad [e_3, e_2] = e_2.$$

For the corresponding left-invariant metrics we have

$$G_{-1} = \text{Sol}, \quad G_0 = \mathcal{H}^2 \times \mathbb{R}, \quad G_1 = \mathcal{H}^3.$$

Proposition 2 *The potentials of the Weierstrass representation for surfaces in G_μ are as follows:*

$$U_\mu = \frac{H}{2} e^\alpha + \frac{\mu+1}{4} |\psi_1|^2 + \frac{\mu-1}{4} \frac{\bar{\psi}_2^2 \bar{\psi}_1}{\psi_1},$$

$$V_\mu = \frac{H}{2} e^\alpha - \frac{\mu+1}{4} |\psi_2|^2 - \frac{\mu-1}{4} \frac{\bar{\psi}_1^2 \bar{\psi}_2}{\psi_2}.$$

The generating spinor ψ of a minimal surface in G_μ meets the equations

$$\bar{\partial} \psi_1 = -\frac{\mu+1}{4} \psi_2^2 \bar{\psi}_2 - \frac{\mu-1}{4} \bar{\psi}_1^2 \bar{\psi}_2,$$

$$\partial \psi_2 = -\frac{\mu+1}{4} \psi_1^2 \bar{\psi}_1 - \frac{\mu-1}{4} \bar{\psi}_2^2 \bar{\psi}_1.$$

In early 1900s for proving the existence of three closed nonselfintersecting geodesics on a two-sphere with a general metric, Poincare proposed to take an analytical μ -parameter family of metrics which joins the metric on the ellipsoid with three different axes and the given metric and then to consider the analytical continuation in μ of the plane sections of the ellipsoid. This program was not realized however it led to some interesting results on perturbations of closed geodesics under deformations of metrics.

It also would be interesting to study the μ -deformations of integrable surfaces in G_μ . Probably that could help to

extend some global results on well-studied minimal or, more general, constant mean curvature surfaces in $G_1 = \mathcal{H}^3$ to such surfaces in Sol.

4. Constant mean curvature (CMC) surfaces in Lie groups

The second fundamental form of a surface in \mathbb{R}^3 is uniquely determined by the mean curvature H and the Hopf quadratic differential

$$Adz^2 = (x_{zz}, N)dz^2,$$

where $x_{zz} = \frac{\partial^2 x}{\partial z^2}$ and N is the unit normal vector field. We have

$$|A|^2 = \frac{(\kappa_1 - \kappa_2)^2 e^{4\alpha}}{16}$$

where κ_1 and κ_2 are the principal curvatures. In terms of ψ this differential takes the form

$$A = \bar{\psi}_2 \partial \psi_1 - \psi_2 \partial \bar{\psi}_2.$$

The Gauss–Codazzi equations are

$$\alpha_{z\bar{z}} + U^2 - |A|^2 e^{-2\alpha} = 0 \tag{9}$$

which is the Gauss formula for the curvature in terms of the metric and

$$A_{\bar{z}} = (U_z - \alpha_z U) e^\alpha$$

which implies that A is holomorphic if and only if $H = \text{const.}$ ⁶

Since the only holomorphic quadratic differential on a sphere vanishes everywhere, any CMC sphere in \mathbb{R}^3 is umbilic, i.e., $\kappa_1 = \kappa_2$ everywhere, and it is easily to derive that any closed umbilic surface is a round sphere. For tori the holomorphic quadratic differentials are constant and, since there are no umbilic tori, the Hopf differential of a CMC torus equals $\text{const} \cdot dz^2 \neq 0$. By a dilation any CMC torus is transformed into the torus with $H = 1$ and then by rescaling a conformal parameter we may achieve $A = \frac{1}{2}$. Then (9) takes the form

$$u_{z\bar{z}} + \sinh u = 0, \quad u = 2\alpha, \tag{10}$$

which is the integrable elliptic sinh-Gordon equation (see the classification of such tori based on this integrable system in [19]).

⁶The analogous results were established by Hopf also for surfaces in other space forms, S^3 and \mathcal{H}^3 .

Recently such an approach was extended for studying CMC surfaces in other ambient spaces. The breakthrough point was a result of Abresch and Rosenberg who proved that

- *there is a generalized Hopf differential $A_{AR}dz^2$ which is defined on any surface in $S^2 \times \mathbb{R}$ or $\mathcal{H}^2 \times \mathbb{R}$ such that for CMC surfaces A_{AR} is holomorphic*

by deriving the explicit formula for this differential [2]. This differential vanishes identically on a CMC sphere and they are shown that if the equations $H = \text{const}$ and $A_{AR} = 0$ are satisfied on a closed surface M then M is a sphere of revolution which implies that

- *every CMC sphere in $S^2 \times \mathbb{R}$ or $\mathcal{H}^2 \times \mathbb{R}$ is a sphere of revolution.*

Later they extended that for surfaces in other homogeneous manifolds with a four-dimensional isometry group [3]. Moreover Abresch announced that

- *only the spaces $E(\kappa, \tau)$ admit generalized Hopf differentials which are holomorphic on CMC surfaces.*

The mashinery of the Weierstrass representation admits us to derive very easily such differentials for surfaces in Nil and $SL(2, \mathbb{R})$ and moreover to study (the first time) the following problem:

When the holomorphicity of the generalized Hopf differential implies that the surface has constant mean curvature?

It appeared that although for Nil the answer is positive as for space forms in general, there are non-CMC surfaces with holomorphic generalized Hopf differential (see [7] and below).

We have

Theorem 2 ([4]) *Let us denote by $Adz^2 = (\nabla_{f_z} f_z, N)dz^2$ the Hopf differential of a surface $f : M \rightarrow G$. Then*

1. *for $G = \text{Nil}$ the quadratic differential*

$$\tilde{A}dz^2 = \left(A + \frac{Z_3^2}{2H + i} \right) dz^2 \quad (11)$$

is holomorphic on a surface if and only if the surface has constant mean curvature;

2. *for $G = \widetilde{SL(2, \mathbb{R})}$ the quadratic differential*

$$\tilde{A}dz^2 = \left(A + \frac{5}{2(H - i)} Z_3^2 \right) dz^2$$

is holomorphic on constant mean curvature surfaces.

The original Abresch–Rosenberg differential A_{AR} derived in [2, 3] is slightly different from ours:

$$A_{\text{AR}} = (H + i\tau)\tilde{A}$$

where τ is the bundle curvature (see footnote on page 1). These differentials behave differently for non-CMC surfaces. Fernandez and Mira [7] showed how the definition of \tilde{A} is extended for other spaces $E(\kappa, \tau)$ and proved that

- a compact surface $M \subset E(\kappa, \tau)$ with holomorphic differential \tilde{A} (if $\tau \neq 0$ we assume that M is not a torus) is a CMC surface;
- in $\mathcal{H}^2 \times \mathbb{R}$ and $\widetilde{SL(2, \mathbb{R})}$ all surfaces with holomorphic differential \tilde{A} are CMC-surfaces or some non-compact surfaces whose complete description is given in [7];
- there are non-compact rotationally-invariant non-CMC surfaces with holomorphic differential A_{AR} in $S^2 \times \mathbb{R}$ and $\mathcal{H}^2 \times \mathbb{R}$ however it is still unclear are there non-CMC surfaces with holomorphic differential \tilde{A} in such ambient spaces.

As we see above CMC-tori in \mathbb{R}^3 are described by the elliptic sinh-Gordon equation. By [4, 7], in the spaces $E(\kappa, \tau)$ except probably some Berger spheres CMC tori are exactly the tori with holomorphic differential \tilde{A} . It appeared that for surfaces in Nil the holomorphicity of \tilde{A} again leads to the elliptic sinh-Gordon equation but for other quantities.

Theorem 3 (Berdinsky) *For a certain choice of a conformal parameter the potential $U = V$ of the Weierstrass representation of a CMC torus has to meet the equation*

$$v_{z\bar{z}} + 2 \sinh 2v = 0 \quad (12)$$

where $v = \log U$.

First we prove the following

Lemma 1 (Berdinsky) *In terms of ψ and of the differential*

$$B = \frac{1}{4}(2H + i)\tilde{A}$$

the derivational equations for surfaces in Nil are written as follows

$$\partial \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \begin{pmatrix} v_z - \frac{1}{2}H_z e^{-v} e^\alpha & B e^{-v} \\ -e^v & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (13)$$

$$\bar{\partial} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \begin{pmatrix} 0 & e^v \\ -\bar{B} e^{-v} & v_{\bar{z}} - \frac{1}{2}H_{\bar{z}} e^{-v} e^\alpha \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad (14)$$

Proof of Lemma. We have

$$\frac{\partial U}{\partial z} = v_z e^v = \frac{2H+i}{4} \psi_2 \partial \bar{\psi}_2 + \frac{2H-i}{4} \bar{\psi}_1 \partial \psi_1 - \frac{iH}{2} \psi_1 \bar{\psi}_2 |\psi_2|^2 + \frac{H_z e^\alpha}{2}$$

and combining that with (11) we yield

$$\partial \psi_1 = (v_z - \frac{1}{2} H_z e^{-v} e^\alpha) \psi_1 + \frac{1}{4} (2H+i) \tilde{A} e^{-v} \psi_2,$$

where $e^\alpha = |\psi_1|^2 + |\psi_2|^2$. Analogous calculations gives us

$$\frac{\partial U}{\partial \bar{z}} = v_{\bar{z}} e^v = \frac{2H+i}{4} \bar{\psi}_2 \bar{\partial} \psi_2 + \frac{2H-i}{4} \psi_1 \bar{\partial} \bar{\psi}_1 - \frac{iH}{2} \psi_2 \bar{\psi}_1 |\psi_1|^2 + \frac{H_{\bar{z}} e^\alpha}{2}$$

and

$$\bar{\partial} \psi_2 = -\frac{1}{4} (2H-i) e^{-v} \tilde{A} \psi_1 + (v_{\bar{z}} - \frac{1}{2} H_{\bar{z}} e^{-v} e^\alpha) \psi_2.$$

Together with the Dirac equation $\mathcal{D}\psi = 0$ these equations constitute (13) and (14). Lemma is proved.

Now let us prove the theorem. We again recall that holomorphic differentials on tori are constant: $\text{const} \cdot dz^2$. CMC surfaces in Nil with $\tilde{A} = 0$ are spheres of revolution [3, 5]. Hence H and \tilde{A} are nonvanishing constants and the equations (13) and (14) are simplified as follows

$$\bar{\partial} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 0 & e^v \\ -\tilde{B} e^{-v} & v_{\bar{z}} - \frac{1}{2} H_{\bar{z}} e^{-v} e^\alpha \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

which implies

$$v_{z\bar{z}} + e^{2v} - |B|^2 e^{-2v} = 0.$$

By rescaling the conformal parameter we achieve that $|B| = 1$. This proves Theorem.

In this case the appearance of the same integrable system as the Gauss–Codazzi equations for different classes of surfaces (CMC tori in \mathbb{R}^3 and in Nil) does not mean any Lawson type correspondence because for tori in Nil this equation is written not on the metric but on the potential U of the Weierstrass representation.⁷ Moreover this coincidence does imply the local isometry of corresponding surfaces.

We would like also to mention that until recently there are no known examples of CMC tori in Nil and this theorem is just a step to proving their existence. One of the main difficulties is that the systems (10) and (12) are very different from the physical point of view: they describe different fields, i.e., the function u in (10) is real-valued and the function v in (12) in general has nontrivial real and imaginary parts. Hence the reality conditions for these systems are drastically different. However it sounds possible to use soliton technique kind of the Lamb ansatz to construct some analogs of the Abresch tori in \mathbb{R}^3 [1].

⁷From the traditional point of view which we do not follow, U is not considered as a geometrical quantity.

5. The spinor energy and the isoperimetric problem [22]

Although in general for surfaces in Nil and $\widetilde{SL(2, \mathbb{R})}$ the potentials U and V are complex-valued, the (spinor) energy functional (5) is real-valued for compact oriented surfaces without boundary. Moreover as in the Euclidean case it is written in geometrical terms:

Theorem 4 ([4]) *For a closed oriented surface M in G its (spinor) energy*

$$E(M) = \int_M UV dx dy$$

equals

$$\begin{aligned} & \frac{1}{4} \int_M \left(H^2 + \frac{\widehat{K}}{4} - \frac{1}{16} \right) d\mu \quad \text{for } G = \text{Nil}; \\ & \frac{1}{4} \int_M \left(H^2 + \frac{5}{16} \widehat{K} - \frac{1}{4} \right) d\mu \quad \text{for } G = \widetilde{SL(2, \mathbb{R})}, \end{aligned}$$

where \widehat{K} is the sectional curvature of the ambient space along the tangent plane to the surface and $d\mu$ is the induced measure.

These expressions for E are different from the Willmore functional which for surfaces in a general ambient space is defined as

$$\mathcal{W} = \int_M (|H|^2 + \widehat{K}) d\mu.$$

For surfaces in \mathbb{R}^3 we have

$$E = \frac{1}{4} \mathcal{W} = \frac{1}{4} \int_M \left(\frac{\kappa_1 + \kappa_2}{2} \right)^2 d\mu = \frac{1}{4} \int_M \left(\frac{\kappa_1 - \kappa_2}{2} \right)^2 d\mu + \frac{1}{4} \int_M \kappa_1 \kappa_2 d\mu.$$

The Gauss–Bonnet theorem implies that for a compact oriented surface M without boundary we have

$$E(M) = \frac{1}{4} \int_M \left(\frac{\kappa_1 - \kappa_2}{2} \right)^2 d\mu + \frac{2\pi\chi(M)}{4} \quad (15)$$

where $\chi(M)$ is the Euler characteristic of M . In particular this implies that for spheres

$$E \geq \pi$$

and the equality is achieved exactly on the round spheres for which $\kappa_1 = \kappa_2$ everywhere.

We note that the round spheres are exactly the isoperimetric profiles in \mathbb{R}^3 , i.e. these are closed surfaces of minimal area among all surfaces bounding domains of some fixed volume. It follows from the variational principle that an isoperimetric profile is always a CMC hypersurface at regular points and it is

known that if the dimension of the ambient space is not greater than seven then an isoperimetric profile is smooth.

The isoperimetric problem is not solved until recently for surfaces in Nil. However it is known that in general for a compact Riemannian manifold for small volumes the isoperimetric profiles are homeomorphic to a sphere [17]. Hence for small volumes the isoperimetric profiles in Nil are CMC spheres. By [2] all CMC spheres are rotationally invariant, and by [9], CMC spheres of revolution form a family parameterized by the mean curvature $H, 0 < H < \infty$. We compute that

Proposition 3 *For CMC spheres in Nil*

1. *the energy functional is constant and equals $E = \pi$;*
2. *the Willmore functional varies as follows:*

$$\mathcal{W}(H) = 10\pi + \frac{\pi}{2H^2} - \pi \frac{(1 + 4H^2)(3H^2 - \frac{1}{4})H^3}{2} \left(\frac{\pi}{2} - \arctan \left[\frac{4H^2 - 1}{4H} \right] \right).$$

Let us consider general surfaces of revolution in Nil. There is the natural submersion

$$\text{Nil} \rightarrow \text{Nil}/SO(2)$$

onto the half-plane $u \geq 0$ with the metric

$$du^2 + \frac{4dv^2}{4 + u^2}.$$

Let $\gamma(s) = (u(s), v(s))$ be a path-length parameterized smooth curve in this halfplane which generates by revolution a surface in Nil. Let us denote by σ the angle between γ and the vector $\frac{\partial}{\partial u}$. We have

Theorem 5 ([5]) *For a closed oriented surface M in Nil obtained by revolving a curve $\gamma \subset B$ around the z -axis, the spinor energy of M equals*

$$\begin{aligned} E(M) &= \frac{1}{4} \int_{\gamma} \left(H^2 - \frac{1}{4} n_3^2 \right) d\mu = \\ &= \frac{\pi}{8} \int_{\gamma} \left(\dot{\sigma} - \frac{\sin \sigma}{u} \right)^2 \sqrt{4u^2 + u^4} ds - \frac{\pi}{4} \int_{\gamma} \frac{\partial [\dot{u} \sqrt{4 + u^2}]}{\partial s} ds = \end{aligned} \quad (16)$$

$$\frac{\pi}{8} \int_{\gamma} \left(\dot{\sigma} - \frac{\sin \sigma}{u} \right)^2 \sqrt{4u^2 + u^4} ds + \frac{\pi \chi(M)}{2}$$

where $\chi(M)$ is the Euler characteristic of M .

If $\dot{\sigma} = \frac{\sin \sigma}{u}$ everywhere then the surface is a CMC sphere.

It implies

Corollary 4 *For spheres of revolution in Nil we have*

$$E(M) \geq \pi$$

and the equality is attained exactly at CMC spheres.

Corollary 5 *For tori of revolution in Nil the spinor energy is positive:*

$$E(M) > 0.$$

It is also straightforward to prove

Proposition 4 ([5]) *The CMC spheres in Nil are the critical points of the spinor energy functional E .*

We see now that except the spectral theory of the Dirac operator there are other reasons to treat the spinor energy as the right analog of the Willmore functional for surfaces in Nil. Indeed,

- it takes the constant value on the CMC spheres which are the critical points of this functional;
- there is a strong similarity of formulas (15) and (16). However the quantities $\dot{\sigma}$ and $\frac{\sin \sigma}{u}$ are not the principal curvatures of a surface of revolution and two poles are the only umbilic points on a CMC sphere in Nil;
- the conditions $A = 0$ and $\tilde{A} = 0$ distinguish in \mathbb{R}^3 and Nil the minima of E for spheres of revolution (in the Euclidean case even for spheres).

Of course, this study has to be completed and the following questions are worth to be answered:

1. *is E bounded from below for each topological type of closed oriented surfaces?*
2. *is E positive?*
3. *are the CMC spheres in Nil are the global minima of E for spheres?*
4. *how to generalize (16) for general surfaces?*
5. *what are the minima of E for surfaces of fixed topological type and, in particular, what is the substitution of the Willmore conjecture?*

It is also interesting to study the analogous questions for surfaces in $\widetilde{SL(2, \mathbb{R})}$ for which the spinor energy functional also has a geometrical form.

For $S^2 \times \mathbb{R}$ we have the following computational observation:

Proposition 5 ([5]) *For isoperimetric profiles M in $S^2 \times \mathbb{R}$ we have*

$$\int_M (H^2 + \widehat{K} + 1) d\mu = 16\pi.$$

The isoperimetric problem for $S^2 \times \mathbb{R}$ was solved by Pedrosa [18] who proved that for volumes $d \leq d_0$ the isoperimetric profiles are CMC spheres, for $d > d_0$ the isoperimetric profiles bound the product cylinders $S^2 \times [0, \frac{d}{4\pi}]$ where d_0 is some transition point from one topological class of solutions to another. The functional mentioned in Proposition takes the same value on all CMC spheres (not only isoperimetric) and on all isoperimetric profiles (connected and disconnected).

We would like to guess that

the right analog of the Willmore theory (at least for spheres) has to be related to the isoperimetric problem and the isoperimetric profiles in three-dimensional homogeneous spaces have to be distinguished as (at least local) minima of the Willmore type functional which is constant on them.

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